Testing for certain idempotent Maltsev conditions

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Near Unanimity

Definition

A function $f(x_0, \ldots, x_{n-1})$ on a set A is called a near unanimity operation on A if the following equations hold:

$$f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x, x) \approx \cdots \approx f(x, x, x, \ldots, x, y) \approx x.$$

A term t of an algebra $\mathbf{A}$ is a near unanimity term for $\mathbf{A}$ if the operation $t^{\mathbf{A}}$ is a near unanimity operation on A.

Remark

Note that a near unanimity term t is idempotent, i.e., it satisfies the equation $t(x, x, \ldots, x) \approx x$. An algebra is idempotent if all of its term operations are idempotent.

Robustness

Question

- One can ask if having a near unanimity term is preserved under varietal joins, i.e., if $\mathcal{V}_0$ and $\mathcal{V}_1$ have near unanimity terms, will $\mathcal{V}_0 \vee \mathcal{V}_1$?
- If $\mathcal{V}_i$ is generated by the algebra $\mathbf{A}_i$, then this amounts to checking if the product $\mathbf{A}_0 \times \mathbf{A}_1$ has a near unanimity term.

Definition

- The **Maltsev product** of varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, denoted $\mathcal{V}_0 \circ \mathcal{V}_1$, is the class of all algebras $\mathbf{A}$ such that for some congruence θ of $\mathbf{A}$, the quotient $\mathbf{A}/\theta$ is in $\mathcal{V}_1$ and for each $a \in A$, $a/\theta \in \mathcal{V}_0$.
- (Freese, McKenzie) A property of varieties is robust if it is preserved under Maltsev products of idempotent varieties.

Robustness of near unanimity

Theorem (Marković, Maróti, McKenzie)

If $\mathcal{V}_0$ and $\mathcal{V}_1$ are idempotent varieties that have near unanimity terms of arities n and m respectively, then $\mathcal{V}_0 \vee \mathcal{V}_1$ and $\mathcal{V}_0 \circ \mathcal{V}_1$ have a near unanimity term of arity nm.

Proof.

If t_i is the near unanimity term for V_i, then the following term is a near unanimity term for $\mathcal{V}_0 \vee \mathcal{V}_1$ and $\mathcal{V}_0 \circ \mathcal{V}_1$:

$$t_0(t_1(x_0, \dots, x_{m-1}), t_1(x_m, \dots, x_{2m-1}), \dots, t_1(x_{(n-1)m}, \dots, x_{nm-1})).$$

□

Question

- Can we do better than nm in the previous theorem?
- Initial Answer: Probably not, but maybe in special cases.

A computational approach

First Try

- Build small idempotent algebras $\mathbf{A}_0$ and $\mathbf{A}_1$, manually, each with 3-ary basic operations p_0 and p_1 such that p_i is a majority term for $\mathbf{A}_i$.
- Use UACalc to test if $\mathbf{A}_0 \times \mathbf{A}_1$ has a small arity near unanimity term, first checking for a majority term.
- Keep going until an example is found that doesn't have a majority term, and then start looking for examples that don't have a k-ary near unanimity term for $k < 9$.

Problems

- The manual approach didn't yield any examples that didn't have a low arity near unanimity term.
- For ease of calculation we considered 2-element algebras.
- The version of UACalc in use at the time used an EXP-time algorithm to check for the presence of a k-ary near unanimity term.

A Polynomial-time algorithm

Theorem (Freese-Valeriote)

A finite idempotent algebra $\mathbf{A}$ *has a majority term if and only if for all a, b, c* $\in A^3$,

$$(a, c) \in (Cg^{\mathbf{B}}(a, b) \wedge Cg^{\mathbf{B}}(a, c)) \circ (Cg^{\mathbf{B}}(b, c) \wedge Cg^{\mathbf{B}}(a, c)),$$

where $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}}(\{a, b, c\})$

Remarks

- *This result can be converted into a polynomial time algorithm to test idempotent algebras for a majority term.*
- *It can be extended to handle checking for k-ary near unanimity terms.*
- *The algorithm isn't difficult to implement in Java, using the UACalc library, but it runs slowly.*
- *This can't be used to quickly build a majority term, if one exists.*

A better algorithm

Definition

An operation $f(x_0, \ldots, x_{n-1})$ on A is a local near unanimity operation on A for a subset $S \subseteq A^2 \times \{0, 1, \ldots n-1\}$ if whenever $(a, b, i) \in S$, then $f(a, a, \ldots, a, b, a, \ldots, a) = a$, where b is substituted for x_i in f.

Theorem (Horowitz, McKenzie)

A finite idempotent algebra $\mathbf{A}$ *has an n-ary near unanimity term if and only if for all subsets S of* $A^2 \times \{0, 1, \ldots n-1\}$ *of size n or less,* $\mathbf{A}$ *has an n-ary term operation that is local for S.*

Corollary

A finite idempotent algebra $\mathbf{A}$ *has an n-ary near unanimity term if and only if for all* $a_i, b_i \in A$, $0 \leq i < n$, *the tuple* $(a_0, a_1, \ldots, a_{n-1})$ *is in the subalgebra of* $\mathbf{A}^n$ *generated by*

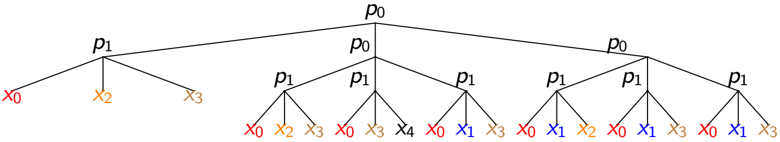
$$\{(a_0, a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{n-1}) : 0 \leq i < n\}.$$

- An implementation of the algorithm based on the Horowitz/McKenzie Theorem was used to check for majority terms in all algebras $\mathbf{A}_0 \times \mathbf{A}_1$, where $\mathbf{A}_0$ and $\mathbf{A}_1$ have universe $\{0, 1\}$ and each has two 3-ary basic operations p_0 and p_1 such that p_i is a majority term for $\mathbf{A}_i$.
- It took less than 5 minutes to find that of the 4096 such pairs, only 28 fail to have a majority term.
- Up to isomorphism, there are only a few (3 or 4) distinct pairs that don't have a majority term.
- These pairs were investigated using the Horowitz/McKenzie algorithm and were found to all have a 5-ary near unanimity term. Each run took less than 5 minutes.
- Fortunately, the 2-generated free algebras in the varieties generated by these pairs were small and so UACalc was able to construct the 5-ary near unanimity terms.

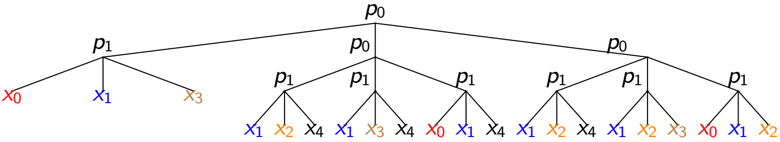
A 5-ary near unanimity term

$t(x_0, x_1, x_2, x_3, x_4) = p_0(t_0, t_1, t_2)$, where

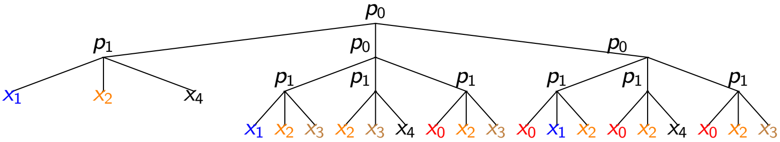
$t_0 =$



$t_1 =$



$t_2 =$



First theorem

Theorem

Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be idempotent algebras that have majority terms p_0 and p_1 respectively. Then the 5-ary term t on the previous slide is a near unanimity term for the algebra $\mathbf{A}_0 \times \mathbf{A}_1$.

Remarks

- *We can't do better than 5-ary in all cases.*
- *The 5-ary term from the theorem doesn't work for Maltsev products of idempotent varieties that have majority terms.*
- *When we move from majority terms to higher arity terms, exhaustive searches aren't feasible.*
- *We found an example of the form $\mathbf{A}_0 \times \mathbf{A}_1$, where each factor has size 2 and has a 4-ary near unanimity term such that, even using the fast near unanimity checker, it would take thousands of hours to check for a 7-ary near unanimity term.*

Extensions

Definition

Let $n,\ m > 2$. Let $P(n, m)$ be

> *the smallest k such $\mathcal{V}_0 \vee \mathcal{V}_1$ will have a k-ary near unanimity term whenever $\mathcal{V}_0$ and $\mathcal{V}_1$ are idempotent varieties that have n-ary and m-ary near unanimity terms, respectively.*

Define $M(n, m)$ similarly, for Maltsev products of varieties.

Remarks

- *We know that $n, m \leq P(n, m) \leq M(n, m) \leq nm$.*
- *The previous theorem can be stated as: $P(3, 3) = 5$.*

Theorem

$n + m - 1 \leq P(n, m) \leq \frac{nm}{2}$.

$n + m - 1 \leq P(n, m)$

- The computation showing that $P(3,3) \neq 4$ provides a pair of 2-element algebras $\mathbf{A}_0$ and $\mathbf{A}_1$, each having majority terms as basic operations, whose product doesn't have a 4-ary near unanimity term.
- The other 3-ary basic operation of $\mathbf{A}_0$ takes on the value 1 only on input $(1,1,1)$, while $\mathbf{A}_1$'s only takes on the value 0 on input $(0,0,0)$.
- Fortunately, this pair can be naturally generalized to show that $P(n, m) > n + m - 2$.
- Let $\mathbf{A}_0$ be the algebra with universe $\{0, 1\}$ and basic operations $p_0^{\mathbf{A}_0}$ and $p_1^{\mathbf{A}_0}$ defined by:

$$p_0^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{n-1}) = \bigwedge_{0 \leq i < j < n} (x_i \vee x_j)$$

$$p_1^{\mathbf{A}_0}(x_0, x_1, \ldots, x_{m-1}) = \bigwedge_{0 \leq i < m} x_i$$

- If $\mathbf{A}_1$ is defined dually then $\mathbf{A}_0 \times \mathbf{A}_1$ fails to have an $n + m - 2$-ary near unanimity term.

$M(3,3)$

The computational approach: Attempt to show $M(3,3) > 5$

- Search through all examples of 4-element idempotent algebras $\mathbf{A}$ that have two 3-ary terms p_0 and p_1 and a congruence θ such that p_1 is majority on $\mathbf{A}/\theta$ and p_0 is majority on the two 2-element θ-classes.
- Try to find examples that don't have a 5-ary near unanimity term.
- The problem with this approach is that there are 2^{114} such algebras.
- A random search of several million algebras from this collection always produced examples with 5-ary near unanimity terms.

The computational approach: Attempt to show $M(3,3) = 5$

- Look for examples that have a 5-ary term, use UACalc to build the term and hope to find a term that works for all Maltsev products.
- Problem: UACalc uses the 2-generated free algebra to build the term, and they were all too big.

$M(n, m) = n + m - 1$

Remarks

- *In light of the computational evidence and the results for* $P(3,3)$*, it seemed reasonable to try to prove that* $M(3,3) = 5$*.*
- *A variation of the term that witnessed* $P(3,3) = 5$ *was found that works for all Maltsev products of idempotent varieties that have majority terms.*
- *Ad hoc techniques were found that were used to show that* $M(n,m) = n + m - 1$ *for small values of* n *and* m*.*

Theorem

Let $\mathcal{V}_0$ *and* $\mathcal{V}_1$ *be idempotent varieties having n-ary and m-ary near unanimity terms, respectively. Then* $\mathcal{V}_0 \circ \mathcal{V}_1$ *has a near unanimity term of arity* $n + m - 1$*.*

Sketch of proof

Definition

Let $d = n + m - 1$ and let S be a subset of the variables $\{x_0, x_1, \ldots, x_{d-1}\}$. A term $t(x_0, x_1, \ldots x_{d-1})$ of arity d is a **near unanimity term for** S if

- t is a near unanimity term for the variety $\mathcal{V}_1$, and
- $\mathcal{V}_0 \circ \mathcal{V}_1$ satisfies the equation $t(x, x, \ldots, x, y, x, \ldots, x) \approx x$ whenever y is substituted in t for any one of the variables x_i from S and x is substituted for all of the other variables of t.

Proof by induction on $|S|$

The following term shows that there is a near unanimity term for the set $S = \{x_0, x_1, \ldots, x_{n-1}\}$:

$$p_0(p_1(x_0, x_n, x_{n+1}, \ldots, x_{n+m-2}), p_1(x_1, x_n, x_{n+1}, \ldots, x_{n+m-2}), \ldots, p_1(x_{n-1}, x_n, x_{n+1}, \ldots, x_{n+m-2})).$$

Sketch of proof

The induction step

- Let $S = \{x_0, x_1, \ldots, x_k\}$ and assume that near unanimity terms exist for all smaller sets of variables.
- For $0 \leq i < n$, let $S_i = \{x_0, x_1, \ldots, x_k\} \setminus \{x_i\}$ and let t_i be a d-ary term that is a near unanimity term for the set S_i.
- The following term is a near unanimity term for S:

$$p_0(t_0(x_0, \ldots, x_{d-1}), t_1(x_0, \ldots, x_{d-1}), ..., t_{n-1}(x_0, \ldots, x_{d-1})).$$

Remark

The term constructed in this proof has depth $m+1$ and it length is $n^m + n^{m-1} + \cdots + n + 1$. For small values of n and m we have been able to construct much shorter, and slightly shallower terms.

n-permutability

Remark

Work by Freese and McKenzie on robustness, along with an interest in finding polynomial-time algorithms to test for idempotent Maltsev conditions, provided the motivation for thinking about the robustness of near unanimity operations and also congruence n-permutability.

Theorem (Valeriote-Willard)

A finite idempotent algebra $\mathbf{A}$ *generates a congruence* $(n+1)$*-permutable variety if and only if for every pair of* $(n+1)$*-tuples* $(a_0, a_1, \ldots, a_n)$, $(b_0, b_1, \ldots, b_n)$ *of elements from* A*, the pair* (a_0, b_n) *is in the relational product* $R_1 \circ R_2 \circ \cdots \circ R_n$*, where* R_i *is the subuniverse of* $\mathbf{A}^2$ *generated by the pairs* (a_{i-1}, a_i), (b_{i-1}, a_i)*, and* (b_{i-1}, b_i).

Corollary

For $n \geq 1$*, there is a poly-time algorithm to decide if a finite idempotent algebra generates a congruence* $(n+1)$*-permutable variety.*

Conclusion

- Useful insights into some problems can be obtained from computational experimentation.
- Only special classes of problems are amenable to this sort of experimentation.
- New insights into poly-time algorithms and implementations of them can arise from this sort of investigation.
- The UACalc library is easy to use and modify.

Problem

Characterize those idempotent Maltsev conditions for which there is a polynomial-time algorithm to determine if a given finite idempotent algebra generates a variety that satisfies it.